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Superfluidity within exact renormalization group approach

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Abstract

The application of the exact renormalization group to a many-fermion system with a short-range attractive force is studied. We assume a simple ansatz for the effective action with effective bosons, describing pairing effects and derive a set of approximate flow equations for the effective coupling including boson and fermionic fluctuations. The phase transition to a phase with broken symmetry is found at a critical value of the running scale. The mean-field results are recovered if boson-loop effects are omitted. The calculations with two different forms of the regulator were shown to lead to similar results.

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(Some figures in this article are in colour only in the electronic version)

There is a growing interest in applying the exact renormalization group (ERG) formalism to few- and many-body systems [1–4] when the underlying interaction is essentially non-perturbative. Regardless of the details all ERG-based approaches share the same distinctive feature, a successive elimination/suppression of some modes, resulting in effective interaction between the remaining degrees of freedom [5]. One specific way of implementing such a procedure is to eliminate modes by applying a momentum-space blocking transformation with some physically motivated cutoff. The effect of varying a cutoff is described by nonlinear ERG evolution equations, which include the effect of the eliminated modes. By solving the ERG equations, one can find a scale dependence of the coupling constants and thus determine a path in the space of Lagrangian functionals. The ERG formalism is closely linked to another approach which has become increasingly popular in both few- and many-body physics, effective field theory (EFT). EFT is also based on a separation of scales, removal of some (mainly high-energy) degrees of freedom and use of the effective degrees of freedom instead of the fundamental ones.

In a sense, EFT and ERG compliment each other. EFT can provide the guidance for fixing the initial conditions and ERG can be used as an analytic method to study the evolution of the system as a function of some generic scale parameter. This is especially useful in the case of truly non-perturbative problems, where there are no small parameters one can expand in. This situation is quite common both in few- and many-body problems. One notes that, although

we will focus on the systems consisting of nucleons, many aspects of the following discussion are relevant for the other types of fermionic systems, especially fermionic atoms in traps.

Probably the most important dynamical feature of the nucleon–nucleon interaction is the unnaturally large scattering length that makes a perturbative expansion meaningless both for free nucleons and for nuclear matter. In addition, the presence of the Fermi momentum signals the appearance of another scale which further complicates the use of any perturbative technique. One notes that the use of the perturbation theory for the two-nucleon system in vacuum can be justified only for the hypothetical case of weakly interacting nucleons with a small scattering length. In medium, however, even a weak attraction between nucleons may lead to the intrinsically non-perturbative phenomenon, the superfluidity, characterized by rearrangement of the ground state and appearance of the gap in the spectrum. The fermions form correlated pairs which, depending on the strength of the interaction, may lead to different physical regimes. The weak coupling regime (BCS phase) corresponds to a pair with the spatial size much larger than the radius of the interaction so that no actual bound two-body subsystem is formed, while in the strong regime corresponding to the Bose–Einstein condensation (BEC) the fermion pairs form compact deeply bound two-body states.

It would be appealing to describe all these regimes by starting at a large initial scale with some EFT motivated effective action and then run the scale down to the physical point where the scale parameter vanishes. Thus, the results will depend on one physical parameter, the scattering length of the nucleon–nucleon interaction in free space.

The ideal tool to treat this problem is provided by a variant of the ERG approach, based on the average effective action (AEA) [6]. The AEA is the generating functional of the one-particle irreducible (1PI) correlation function in the presence of an infrared cutoff scale k. Only fluctuations with momenta larger than k are taken into account. For $k \to 0$, all fluctuations are included and we arrive at standard effective action from which all physical correlation functions can be extracted. The evolution equation for the AEA has the following one-loop form:

$$\partial_k \Gamma = -\frac{\mathrm{i}}{2}\,\mathrm{Tr}[(\partial_k R)(\Gamma^{(2)} - R)^{-1}]. \tag{1}$$

Here $\Gamma^{(2)}$ is the second functional derivative of the AEA taken with respect to all types of field included in the action and R is a regulator which should suppress the contributions of states with momenta less than or of the order of running scale k. To recover the full effective action we require $R(k)$ to vanish as $k \to 0$, in other respects its form is rather arbitrary. The concrete functional form of the regulator has no effect on physical results provided no approximations/truncations were made. In practice, however, approximations/truncations are always required to render the system of the evolution equations finite and solvable. Therefore, some dependence on the functional form of regulator is inevitable. The simplest way to estimate this dependence is to solve the system of evolution equations using several choices of regulators. Of course, it does not guarantee the fully quantitative estimate of the errors introduced but may at least give an idea about the size of the corresponding uncertainty.

1. Ansatz for Γ

We demand that at high scale our theory be a purely fermionic theory with the contact interaction described by the Lagrangian

$$\mathcal{L}_i = -\tfrac{1}{4} C_0 (\psi^\dagger \sigma_2 \psi^{\dagger \mathrm{T}})(\psi^{\mathrm{T}} \sigma_2 \psi). \tag{2}$$

Since we are interested in the appearance of the correlated fermion pairs in a physical ground state, we need to parametrize our effective action in a way that can describe the qualitative

change in the physics when this occurs. A natural way to do this is to introduce a boson field whose vacuum expectation value (VEV) describes this correlated pair [7] and study the evolution of this effective degrees of freedom. At the start of the RG evolution, the boson field is not dynamical and is introduced through a Hubbard–Stratonovich transformation of the four-point interaction. As we integrate more and more of the fermion degrees of freedom by running k to lower values, we generate dynamical terms in the bosonic effective action. We note that a somewhat related problem of the evolution of the bosonic condensate in a relativistic scalar model (without fermions) was considered in [8]. In this paper, we treat a single species of fermion. The corresponding ansatz for the boson–fermion effective action can be written as

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger,\mu,k]=\int \mathrm{d}^4x\left[\phi^\dagger(x)\left(Z_\phi(\mathrm{i}\partial_t+2\mu)+\frac{Z_m}{2m}\nabla^2\right)\phi(x)-U(\phi,\phi^\dagger)\right.$$
$$\left.+\psi^\dagger\left(Z_\psi(\mathrm{i}\partial_t+\mu)+\frac{Z_M}{2M}\nabla^2\right)\psi-g\left(\frac{\mathrm{i}}{2}\psi^{\mathrm{T}}\sigma_2\psi\phi^\dagger-\frac{\mathrm{i}}{2}\psi^\dagger\sigma_2\psi^{\dagger\mathrm{T}}\phi\right)\right].$$

Here M is the mass of the fermions in vacuum and the factor $1/2m$ in the boson kinetic term is chosen simply to make Z_m dimensionless. The couplings, the chemical potential μ, the wavefunction renormalizations $Z_{\phi,\psi}$ and the kinetic-mass renormalizations $Z_{m,M}$ all run with k, the scale of the regulator. The bosons are, in principle coupled to the chemical potential via a quadratic term in ϕ, but this can be absorbed into the potential by defining $\bar{U}=U-2\mu Z_\phi\phi^\dagger\phi$. We expand this potential about its minimum, $\phi^\dagger\phi=\rho_0$, so that the coefficients u_i are defined at $\rho=\rho_0$ as

$$\bar{U}(\rho)=u_0+u_1(\rho-\rho_0)+\tfrac{1}{2}u_2(\rho-\rho_0)^2+\tfrac{1}{6}u_3(\rho-\rho_0)^3+\cdots, \tag{4}$$

where we have introduced $\rho=\phi^\dagger\phi$. The phase of the system is determined by the coefficient u_1. In the symmetric phase, we have $\rho_0=0$ so that the expansion takes the form

$$\bar{U}(\rho)=u_0+u_1\rho+\tfrac{1}{2}u_2\rho^2+\cdots. \tag{5}$$

The potential in the condensed phase can be simplified to

$$\bar{U}(\rho)=u_0+\tfrac{1}{2}u_2(\rho-\rho_0)^2+\cdots. \tag{6}$$

In our current work, we shall truncate this potential at quartic order in the field (order ρ^2). However, the fact that we define our coupling constants at the minimum of the potential does mean that we need to consider the next term in the expansion. This will allow us to treat the implicit dependence of the coefficients on ρ_0. We treat the wavefunction renormalization factor for the bosons in the same way, expanding it about $\rho=\rho_0$ as

$$Z_\phi(\rho)=z_{\phi0}+z_{\phi1}(\rho-\rho_0)+\cdots. \tag{7}$$

The other couplings and renormalization factors can be treated similarly.

The fermions are not dressed at this point and the bosons are just auxiliary fields and so we can assume that $Z_\psi(K)=1$, $Z_M(K)=1$.

We have a choice between following the evolution for fixed chemical potential, and allowing μ to run so that the fermion density is kept fixed. The region corresponding to a Bose–Einstein condensate (BEC) of tightly bound pairs corresponds to negative values of μ and is not accessible to evolution at fixed chemical potential. We therefore follow the evolution at fixed density, n. In this context, it is convenient to define the Fermi momentum, p_F, corresponding to this density and the chemical potential at the starting scale by

$$p_F=(3\pi^2n)^{1/3},\qquad \mu(K)=\frac{1}{2M}p_F^2. \tag{8}$$

This connection between μ and p_F only holds in the symmetric phase, where μ does not run.

2. Evolution equations: general structure

In this section, we consider the general structure of the evolution equations. We start by considering the simpler case of evolution at constant chemical potential. The boson potential $\bar{U}$ is obtained by evaluating the effective action for uniform boson fields. It is given by

$$\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4}\partial_k \Gamma, \tag{9}$$

where $\mathcal{V}_4$ is the volume of spacetime. Substituting our expansion of $\bar{U}$, equation (4), on the left-hand side leads to a set of ordinary differential equations for u_n. We choose to evolve following the minimum of the potential and use the expansion around the minimum $\rho = \rho_0(k)$ to define u_n, as in equation (6). In the symmetric phase, $\rho = 0$, it gives the following set of equations:

$$\frac{\mathrm{d}u_n}{\mathrm{d}k} = \frac{\partial^n}{\partial\rho^n}(\partial_k \bar{U})\bigg|_{\rho=0}. \tag{10}$$

In the condensed phase, we need to define the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}k} = \partial_k + \frac{\mathrm{d}\rho_0}{\mathrm{d}k}\frac{\partial}{\partial\rho_0}. \tag{11}$$

Acting on $\partial^n \bar{U}/\partial\rho^n$ with this (and taking the higher derivative term over to the lhs) gives a set of equations, each of which is coupled to the coefficient of the next term through the evolution of ρ_0:

$$\frac{\mathrm{d}u_n}{\mathrm{d}k} - u_{n+1}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} = \frac{\partial^n}{\partial\rho^n}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}. \tag{12}$$

The coefficient u_1 is special since it vanishes for the expansion around the minimum. Imposing the condition $u_1(k) = 0$ gives an equation for the evolution of ρ_0:

$$-u_2\frac{\mathrm{d}\rho_0}{\mathrm{d}k} = \frac{\partial}{\partial\rho}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}. \tag{13}$$

The coefficient u_0 provides information on the energy density of matter. It satisfies the equation

$$\frac{\mathrm{d}u_0}{\mathrm{d}k} = \partial_k \bar{U}\bigg|_{\rho=\rho_0}, \tag{14}$$

which does not couple back into the other equations.

In the condensed phase, we could truncate our potential at quadratic order, as we do on the right-hand sides of these equations, and simply set $u_3 = 0$ on the left-hand side of the equation for u_2. However, a better approximation can be obtained by substituting the exact form for $u_3(k)$ taken from the evolution with fermion loops only, as described below. This is the approach we adopt here. It has the benefit of providing an approximation to $u_2(k)$ and $\rho_0(k)$ that becomes exact in situations where boson loops can be neglected.

For the boson wavefunction renormalization factor, Z_ϕ, we need to consider a time-dependent background field taken in the form like $\phi = \phi_0 + \eta\exp(-\mathrm{i}p_0 t)$, where η is a constant. The evolution of Z_ϕ can then be obtained from

$$\partial_k Z_\phi = \frac{1}{\mathcal{V}_4}\frac{\partial}{\partial p_0}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p_0=0}. \tag{15}$$

If we substitute our expansion, equation (7), we get another set of coupled equations in the condensed phase. Only the first of these is of interest within our current truncation

$$\frac{\mathrm{d}z_{\phi 0}}{\mathrm{d}k} - z_{\phi 1}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} = \frac{1}{\mathcal{V}_4}\frac{\partial}{\partial p_0}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p_0=0,\rho=\rho_0}. \tag{16}$$

Again, $z_{\phi 1}$ corresponds to a term beyond our current level of truncation and so we will take the result from fermion loops only. Making use of the relation between U and $\bar{U}$ we can also deduce the evolution of Z_ϕ from

$$\partial_k Z_\phi = -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}(\partial_k \bar{U}), \tag{17}$$

which gives

$$\frac{\mathrm{d}z_{\phi 0}}{\mathrm{d}k} - z_{\phi 1}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} = -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}. \tag{18}$$

The evolution equations for the other couplings (Z_m, Z_M, Z_ψ, Z_g) can be derived in a similar manner. However, in this paper we allow to run only Z_ϕ parameters in the potential and chemical potential since this is the minimal set needed to include the effective boson dynamics and study the BCS–BEC crossover.

The fermion number density is given by

$$n = -\frac{\partial \bar{U}}{\partial\mu}\bigg|_{\rho=\rho_0}. \tag{19}$$

The evolution equation for n can be written as

$$\frac{\mathrm{d}n}{\mathrm{d}k} - 2z_{\phi 0}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} = -\frac{\partial}{\partial\mu}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}. \tag{20}$$

The equations constructed so far describe the evolution at constant μ. If we want to follow the evolution at a constant density we must allow μ to run with k. In this case, we define the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}k} = \partial_k + \frac{\mathrm{d}\rho_0}{\mathrm{d}k}\frac{\partial}{\partial\rho_0} + \frac{\mathrm{d}\mu}{\mathrm{d}k}\frac{\partial}{\partial\mu}. \tag{21}$$

Applying this to $\partial\bar{U}/\partial\mu$ at $\rho = \rho_0$ gives the evolution equation for n:

$$\frac{\mathrm{d}n}{\mathrm{d}k} - 2z_{\phi 0}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + \chi\frac{\mathrm{d}\mu}{\mathrm{d}k} = -\frac{\partial}{\partial\mu}(\partial_k \bar{U})|_{\rho=\rho_0}, \tag{22}$$

where we have introduced the fermion-number susceptibility

$$\chi = \frac{\partial^2 \bar{U}}{\partial\mu^2}\bigg|_{\rho=\rho_0}. \tag{23}$$

If n is kept constant ($\mathrm{d}n/\mathrm{d}k = 0$) this becomes

$$-2z_{\phi 0}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + \chi\frac{\mathrm{d}\mu}{\mathrm{d}k} = -\frac{\partial}{\partial\mu}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}. \tag{24}$$

This equation describes the evolution of the chemical potential in the broken phase.

The remaining set of evolution equations to be solved (in the broken phase) is

$$\frac{\mathrm{d}u_0}{\mathrm{d}k} + n\frac{\mathrm{d}\mu}{\mathrm{d}k} = \partial_k \bar{U}\bigg|_{\rho=\rho_0}, \tag{25}$$

$$-u_2\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + 2z_{\phi 0}\frac{\mathrm{d}\mu}{\mathrm{d}k} = \frac{\partial}{\partial\rho}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}, \tag{26}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}k} - u_3\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + 2z_{\phi 1}\frac{\mathrm{d}\mu}{\mathrm{d}k} = \frac{\partial^2}{\partial\rho^2}(\partial_k \bar{U})\bigg|_{\rho=\rho_0}, \tag{27}$$

$$\frac{\mathrm{d}z_{\phi 0}}{\mathrm{d}k} - z_{\phi 1}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + \frac{1}{2}\chi'\frac{\mathrm{d}\mu}{\mathrm{d}k} = -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}(\partial_k\bar{U})\bigg|_{\rho=\rho_0}, \tag{28}$$

$$\frac{\mathrm{d}z_{m0}}{\mathrm{d}k} - z_{m1}\frac{\mathrm{d}\rho_0}{\mathrm{d}k} + \alpha_m\frac{\mathrm{d}\mu}{\mathrm{d}k} = -\frac{1}{\mathcal{V}_4}\,\frac{\partial}{\partial(p^2)}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p^2=0,\rho=\rho_0}, \tag{29}$$

where we have defined

$$\chi' = \frac{\partial^3\bar{U}}{\partial\mu^2\partial\rho}\bigg|_{\rho=\rho_0}, \qquad \alpha_m = \frac{1}{\mathcal{V}_4}\,\frac{\partial^2}{\partial\mu\partial(p^2)}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p^2=0,\rho=\rho_0}. \tag{30}$$

The set of evolution equations in the symmetric phase can easily be recovered using the fact that chemical potential does not run in the symmetric phase and that $\rho_0 = 0$. The left-hand sides of these equations contain a number of coefficients that lie beyond our current level of truncation, such as χ, u_3 and $z_{\phi 1}$. We propose to replace these by their exact expressions obtained from evolution with fermion loops only. The formal derivation of the corresponding expressions will be considered in more detail below.

3. Choice of cutoff

In the bosonic sector, we take the regulator to be an additional quadratic term, proportional to $\phi^\dagger(x)\phi(x')$. In the representation used to write the second derivatives above, it has the matrix structure

$$\mathbf{R}_B(q,k) = \begin{pmatrix} R_B(q,k) & 0 \\ 0 & R_B(q,k) \end{pmatrix}, \qquad R_B(q,k) = \frac{k^2}{2m}f(q/k), \tag{31}$$

where $f(x) \to 1$ as $x \to 0$.

In the fermion case, our regulator should be positive for particle states ($Z_M q^2/2M > Z_\psi\mu$) and negative for hole states ($Z_M q^2/2M < Z_\psi\mu$). It should suppress the contributions of states with energies near μ. One easy way to ensure this would be to use the off-diagonal regulator and so generates an artificial gap in the fermion spectrum around μ. However, such a regulator could not be used without a Fermi sea and so would not allow us to connect our results in matter to the interaction between the fermions in vacuum. We therefore choose our regulator to have the structure

$$\mathbf{R}_F(q,\,p_F,k) = \begin{pmatrix} \mathrm{sgn}(q-p_\mu)R_F(q,\,p_F,k) & 0 \\ 0 & -\mathrm{sgn}(q-p_\mu)R_F(q,\,p_F,k) \end{pmatrix}, \tag{32}$$

where we have introduced

$$p_\mu = \sqrt{\frac{Z_\psi 2M\mu}{Z_M}}, \tag{33}$$

the Fermi momentum corresponding to the (running) value of μ.

The function $R_F(q,\,p_F,k)$ should suppress the contributions of states with momenta near the Fermi surface, $|q-p_F| \lesssim k$. Once a large gap has appeared in the fermion spectrum, there are no low-energy fermion excitations and so the fermionic regulator plays little further role. However, while the gap is zero or small, it is crucial that the sign of the regulator matches that of the energy, $Z_M q^2/2M - Z_\psi\mu$, and hence it is μ which appears in the sign functions.

In order to be sure that we are matching onto the same bare NN interaction at the starting scale K, the fermionic regulator should satisfy $R_F(q,\,p_F,K) \simeq R_F(q,0,K)$.

4. Driving terms: potential

In this section, we derive the evolution equations for the parameters of the potential. Calculating the fermion propagator, multiplying by $\partial_k \mathbf{R}_F$ and taking the matrix trace give

$$\frac{1}{2}\operatorname{tr}\left[(\partial_k \mathbf{R}_F)\left(\mathbf{\Gamma}^{(2)}_{FF} - \mathbf{R}_F\right)^{-1}\right] = \frac{2E_{FR}(q,\, p_F,\, k)\,\operatorname{sgn}(q - p_\mu)\partial_k R_F(q,\, p_F,\, k)}{Z_\psi^2 q_0^2 - E_{FR}(q,\, p_F,\, k)^2 - \Delta^2 + \mathrm{i}\epsilon}, \tag{34}$$

where

$$E_{FR}(q,\, p_F,\, k) = \frac{Z_M}{2M}q^2 - Z_\psi \mu + R_F(q,\, p_F,\, k)\,\operatorname{sgn}(q - p_\mu), \qquad \Delta^2 = g^2\phi^\dagger\phi. \tag{35}$$

The poles in this propagator occur at

$$q_0 = \pm\frac{1}{Z_\psi}\sqrt{E_{FR}(q,\, p_F,\, k)^2 + \Delta^2}. \tag{36}$$

At $k = 0$ ($R_F = 0$) in the condensed phase, these become

$$q_0 = \pm\frac{1}{Z_\psi}\sqrt{\left(\frac{Z_M}{2M}\left(q^2 - p_F^2\right)\right)^2 + \Delta^2}, \tag{37}$$

and so the gap in the fermion spectrum at $q = p_F$ is $2\Delta/Z_\psi$.

The corresponding boson matrix trace can be worked out in a similar way. Putting everything together gives

$$\begin{aligned}\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4}\partial_k\Gamma = &-\frac{1}{Z_\psi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{E_{FR}}{\sqrt{E_{FR}^2 + \Delta^2}}\operatorname{sgn}(q - p_\mu)\partial_k R_F \\ &+ \frac{1}{2Z_\phi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{E_{BR}}{\sqrt{E_{BR}^2 - V_B^2}}\partial_k R_B,\end{aligned} \tag{38}$$

where

$$E_{BR}(q,\, k) = \frac{Z_m}{2m}q^2 + u_1 + u_2(2\phi^\dagger\phi - \rho_0) + R_B(q,\, k), \qquad V_B = u_2\phi^\dagger\phi. \tag{39}$$

The driving terms in the evolution equations for the coefficients in our expansion, equation (4), are obtained from the derivatives of $\partial_k \bar{U}$ with respect to $\rho = \phi^\dagger\phi$ so that we get in the condensed phase

$$\begin{aligned}\frac{\partial}{\partial\rho}(\partial_k \bar{U})|_{\rho=\rho_0} = &\frac{g^2}{2Z_\psi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{E_{FR}}{\left(E_{FR}^2 + \Delta^{(c)2}\right)^{3/2}}\operatorname{sgn}(q - p_\mu)\partial_k R_F \\ &+ \frac{u_2 V_B^{(c)}}{2Z_\phi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{E_{BR}^{(c)} - 2V_B^{(c)}}{\left(E_{BR}^{(c)2} - V_B^{(c)2}\right)^{3/2}}\partial_k R_B,\end{aligned} \tag{40}$$

where

$$E_{BR}^{(c)}(q) = \frac{Z_m}{2m}q^2 + u_2\rho_0 + R_B(q,\, k), \qquad V_B^{(c)} = u_2\rho_0, \quad \Delta^{(c)} = g\sqrt{\rho_0}, \tag{41}$$

and similarly for the second derivative of $\partial_k \bar{U}$ with respect to $\rho = \phi^\dagger\phi$. The equations for the couplings $u_{1(2)}$ in both phases can be obtained in a similar way from the driving terms recalling that in the symmetric phase $\rho = \rho_0, u_1 \neq 0$. The driving term for the evolution of the fermion number density is given by the derivative of $\bar{U}$ with respect to μ. The sign functions in the fermion part of equation (38) depend on μ and so, in principle, differentiating with respect to μ could generate surface terms. However, the sign change occurs at precisely

the point where E_{FR} vanishes and hence such terms do not arise. The resulting driving term vanishes in the symmetric phase, and so evolution at constants n and μ is the same there.

For the evolution of μ in the condensed phase, we get

$$-\frac{\partial}{\partial\mu}(\partial_k\bar{U})\bigg|_{\rho=\rho_0} = -\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{\Delta^{(c)2}}{\left(E_{FR}^2+\Delta^{(c)2}\right)^{3/2}}\,\mathrm{sgn}(q-p_\mu)\partial_k R_F - \int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{V_B^{(c)2}}{\left(E_{BR}^{(c)2}-V_B^{(c)2}\right)^{3/2}}\partial_k R_B. \tag{42}$$

5. Driving terms: Z_ϕ

For the wavefunction renormalization factor, Z_ϕ, we need to consider a time-dependent background field. The evolution of Z_ϕ is then driven by

$$\frac{1}{\mathcal{V}_4}\frac{\partial}{\partial p_0}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p_0=0}. \tag{43}$$

Defining

$$\boldsymbol{\Gamma}^{(3)}_{BB\phi} = \frac{\partial}{\partial\phi}\boldsymbol{\Gamma}^{(2)}_{BB} = \begin{pmatrix}-2u_2\phi^\dagger & -2u_2\phi\\ 0 & -2u_2\phi^\dagger\end{pmatrix}, \tag{44}$$

and

$$\boldsymbol{\Gamma}^{(3)}_{FF\phi} = \frac{\partial}{\partial\phi}\boldsymbol{\Gamma}^{(2)}_{FF} = \begin{pmatrix}0 & \mathrm{i}g\sigma_2\\ 0 & 0\end{pmatrix}, \tag{45}$$

we can write the relevant part of the evolution equation in the form

$$\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\bigg|_{\eta=0} = +\mathrm{i}\,\mathrm{Tr}\left[(\partial_k\mathbf{R}_F)\left(\boldsymbol{\Gamma}^{(2)}_{FF}-\mathbf{R}_F\right)^{-1}\boldsymbol{\Gamma}^{(3)\dagger}_{FF\phi}\left(\boldsymbol{\Gamma}^{(2)}_{FF}-\mathbf{R}_F\right)^{-1}\boldsymbol{\Gamma}^{(3)}_{FF\phi}\left(\boldsymbol{\Gamma}^{(2)}_{FF}-\mathbf{R}_F\right)^{-1}\right] - \mathrm{i}\,\mathrm{Tr}\left[(\partial_k\mathbf{R}_B)\left(\boldsymbol{\Gamma}^{(2)}_{BB}-\mathbf{R}_B\right)^{-1}\boldsymbol{\Gamma}^{(3)\dagger}_{BB\phi}\left(\boldsymbol{\Gamma}^{(2)}_{BB}-\mathbf{R}_B\right)^{-1}\boldsymbol{\Gamma}^{(3)}_{BB\phi}\left(\boldsymbol{\Gamma}^{(2)}_{BB}-\mathbf{R}_B\right)^{-1}\right]. \tag{46}$$

After lengthy algebra, one can obtain in broken phase

$$\frac{1}{\mathcal{V}_4}\frac{\partial}{\partial p_0}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\partial_k\Gamma\right)_{\eta=0}\bigg|_{p_0=0} = -\frac{g^2}{4}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{2E_{FR}^2-\Delta^{(c)2}}{\left(E_{FR}^2+\Delta^{(c)2}\right)^{5/2}}\,\mathrm{sgn}(q-p_\mu)\partial_k R_F - \frac{u_2V_B^{(c)}}{2}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{2E_{BR}^{(c)2}-6E_{BR}^{(c)}V_B^{(c)}+V_B^{(c)2}}{\left(E_{BR}^{(c)2}-V_B^{(c)2}\right)^{5/2}}\partial_k R_B. \tag{47}$$

The corresponding expression in the symmetric phase follows rather trivially.

As a check on this result, we note that u_1 contains a piece $-2\mu Z_\phi$. Hence, we can also obtain the evolution of Z_ϕ from

$$-\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}(\partial_k\bar{U})\bigg|_{\rho=\rho_0}. \tag{48}$$

Taking the partial derivative with respect to μ, as discussed at the end of the previous section, the result we obtain agrees with equation (47).

6. Fermions only

In order to estimate the effects of the higher order coefficients (u_3, etc), we use the ERG equations when boson loops are neglected. In this case, the expressions simplify considerably and the total effective potential can be calculated analytically. All the needed coefficients can then be extracted by simple differentiation. The RG equation for the effective potential becomes

$$\partial_k \bar{U} = -\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{E_{FR}}{\sqrt{E_{FR}^2 + \Delta^2}} \operatorname{sgn}(q - p_\mu)\partial_k R_F. \tag{49}$$

(In the absence of boson fluctuations, we have $Z_\psi = 1$.) This can be rewritten in the form

$$\partial_k \bar{U} = -\partial_k \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \sqrt{E_{FR}^2 + \Delta^2}, \tag{50}$$

and immediately integrated to give

$$\bar{U}(\rho, \mu, k) = \bar{U}(\rho, \mu, K) - \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \left[\sqrt{E_{FR}(q, p_F, k)^2 + \Delta^2} - \sqrt{E_{FR}(q, p_F, K)^2 + \Delta^2}\right]. \tag{51}$$

We have made explicit the dependence of the potential on the chemical potential μ.

At our starting scale K, we take the potential to have the form

$$\bar{U}(\rho, \mu, K) = u_0(K) + u_1(K)\rho. \tag{52}$$

The renormalized value of $u_1(K)$ can be deduced from the scattering length. To determine $u_0(K)$, we use the fact that at $\rho = 0$ the physical potential is just that of a free Fermi gas, measured relative to the chemical potential

$$\bar{U}(0, \mu, 0) = 2\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{FR}(q, p_F, 0)\theta(p_F - q), \tag{53}$$

and hence

$$u_0(K) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{FR}(q, p_F, 0) - \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{FR}(q, p_F, K)\operatorname{sgn}(q - p_F). \tag{54}$$

The physical potential (at $k = 0$) is then given by

$$\bar{U}(\rho, \mu, 0) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} [E_{FR}(q, p_F, 0) - \sqrt{E_{FR}(q, p_F, 0)^2 + \Delta^2}] - \frac{M\Delta^2}{4\pi a} + \frac{\Delta^2}{2}\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{E_{FR}(q, 0, 0)}. \tag{55}$$

Differentiating with respect to ρ and setting the derivative equal to zero, we find that Δ^2 at the minimum satisfies

$$-\frac{M}{4\pi a} + \frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \left[\frac{1}{E_{FR}(q, 0, 0)} - \frac{1}{\sqrt{E_{FR}(q, p_F, 0)^2 + \Delta^2}}\right] = 0. \tag{56}$$

This is exactly the gap equation used in [9].

To get the number density of fermions, we can differentiate $\bar{U}(\rho, \mu, 0)$ with respect to μ. This gives

$$n = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \left[1 - \frac{E_{FR}(q, p_F, 0)}{\sqrt{E_{FR}(q, p_F, 0)^2 + \Delta^2}}\right], \tag{57}$$

in agreement with [9].

7. Initial conditions

We start our evolution at some large, fixed value for the cutoff scale, K, and evolve down to $k = 0$. This clearly requires some initial conditions on the parameters in our effective action. Some constraints on these have already been mentioned in section 1, but we now turn to fixing specific values for them. These should be derived from the known interactions between the particles in vacuum. In the vacuum case, it is relatively straightforward to fix the initial values for the parameters at the starting scale K.

In matter, the determination of our renormalized parameters is complicated by the fact that our fermionic cutoff depends on the Fermi momentum. The values of parameters such as $u_1(p_F, K)$ must thus depend on p_F as well as on K. Note that we assume that our starting scale K is sufficiently large that all physical effects of the Fermi sea have been completely suppressed by our cutoff. Hence, the dependence of the initial parameters on p_F is merely to compensate for the p_F dependence of our regularization and renormalization procedures. For example, in the case of a simple sharp cutoff, the maximum momentum included is $K + p_F$ in matter instead of K. It is thus particularly important to correctly renormalize $u_1(p_F, K)$, since this quantity cancels a linear divergence and so can be shifted by a finite amount even for $K \to \infty$.

One could introduce a cutoff function that tends to a p_F-independent form for $K \gg p_F$. However in practice a modification of the renormalization procedure is more convenient. In the region $K \gg p_F$, we can ignore boson loops. The evolution of quantities such as $u_1(p_F, K)$, $u_2(p_F, K)$, $Z_\phi(p_F, K)$ and $Z_m(p_F, K)$ is thus similar to the vacuum case, except for the different cutoffs. This allow us to define $u_1(p_F, K)$ as

$$\frac{u_1(p_F, K)}{g^2} = -\frac{M}{4\pi a} + \frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\left[\frac{1}{E_{FR}(q, 0, 0)} - \frac{\mathrm{sgn}(q - p_F)}{E_{FR}(q, p_F, K)}\right]. \tag{58}$$

This expression can be thought of as being generated by the vacuum evolution using a modified cutoff that interpolates smoothly between $R_F(q, p_F, k)$ for $k \gg p_F$ and $R_F(q, 0, k)$ for $k \lesssim p_F$. It ensures that our renormalized parameter $u_1(p_F, K)$, defined using $R_F(q, p_F, k)$ for large k, corresponds to the physical scattering length in vacuum.

The initial values for $u_2(p_F, K)$, $Z_\phi(p_F, K)$ and $Z_m(p_F, K)$ can be determined using similar procedures, although this is not so crucial since these quantities do not contain linearly divergent pieces and so all their p_F-dependence is suppressed by the powers of p_F/K. One convenient choice is to take their starting values to be zero at some large but finite scale K. An alternative is to require that they tend to zero as $K \to \infty$.

The initial condition for the energy density is most conveniently expressed in terms of $\tilde{u}_0$ which, in the symmetric phase, is simply given by the energy of a free Fermi gas, measured relative to the chemical potential, and so its initial value is just

$$\tilde{u}_0(K) = 2\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{FR}(q, p_F, 0)\theta(p_F - q). \tag{59}$$

8. Results and conclusion

We solve the evolution equations (equations (24) and (26)–(28)) numerically with two types of cutoffs. First, we use the smoothed step-function type of regulator (called hereafter as R_1):

$$R_{1F} = \frac{k^2}{2M}\theta_1(q - p_F, k, \sigma), \qquad R_{1B} = \frac{k^2}{2m}\theta_1(q, k, \sigma), \tag{60}$$

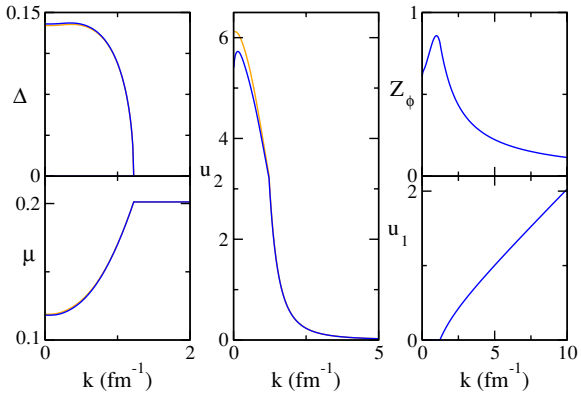

Figure 1. Numerical solutions to the evolution equations for infinite a_0 and $p_F = 1.37$ fm^{-1}, starting from $K = 16$ fm^{-1}. We show the evolution of all relevant parameters for the cases of fermion loops only (thin lines), and of bosonic loops with a running Z_ϕ (thick lines). All quantities are expressed in the appropriate powers of fm^{-1}.

where

$$\theta_1(q,k,\sigma) = \frac{1}{2\mathbf{Erf}(1/\sigma)}\left[\mathbf{Erf}\left(\frac{q+k}{k\sigma}\right) + \mathbf{Erf}\left(\frac{q-k}{k\sigma}\right)\right] \tag{61}$$

with σ being a parameter determining the sharpness of the step.

Second, we use a sharp cutoff function denoted as R_2 chosen to make the calculations as simple as possible:

$$R_{2F} = \frac{1}{2M}\left[((k+p_\mu)^2 - q^2)\theta(p_\mu + k - q) + \left((k+p_\mu)^2 + q^2 - 2p_\mu^2\right)\theta(q - p_\mu + k)\right], \tag{62}$$

$$R_{2B} = \frac{1}{2m}(k^2 - q^2)\theta(k-q). \tag{63}$$

Similar boson regulator was used in [10] (see also [11]).

As we can see the fermion sharp cutoff consists of two terms which result in the modification of the particle and hole propagators respectively. The hole term is further modified to suppress the contribution from the surface terms, which may bring in the dangerous dependence of the regulator on the cutoff scale even at the vanishingly small k. As an example, we focus on the parameters relevant to neutron matter: $M = 4.76\,\text{fm}^{-1}$, $p_F = 1.37\,\text{fm}^{-1}$.

We first discuss the results obtained with the smooth cutoff R_1. We find that the value of the physical gap is practically independent of either the values of the width parameter σ (varied within some range) or the starting scale K provided $K > 5\,\text{fm}^{-1}$. The results of the calculations are shown in figure 1. At the starting scale, the system is in the symmetric phase and remains in this phase until u_1 hits zero at $k_{\text{crit}} \simeq 1.2\,\text{fm}^{-1}$, where the artificial second-order phase transition to a broken phase occurs and the energy gap is formed. Already at $k \simeq 0.5$ the running scale has essentially no effect on the gap. We found very small (in the level of 1%) contribution to the gap from the boson loops, due to cancellations between the direct contributions to the running of the gap and indirect ones via u_2. The boson loops play much more important role in the evolution of u_2 and Z_ϕ. In fact, they drive both couplings to zero at $k \to 0$ although at rather slow pace. We note, however, that the effect of the boson loops for the gap may still be more visible if the evolution of the other couplings is included. The results

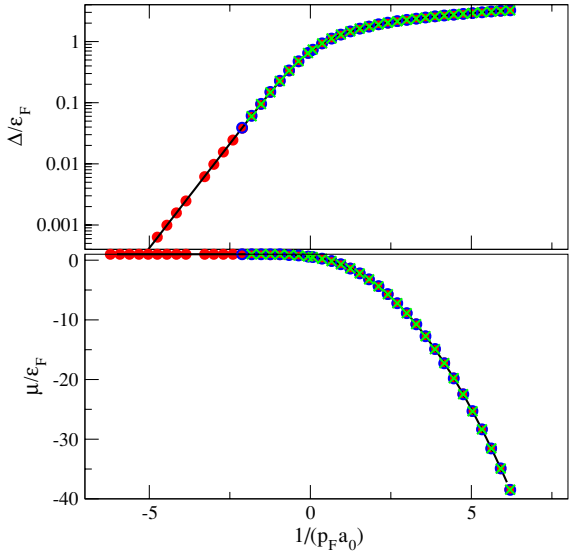

Figure 2. Evolution of the chemical potential.

obtained for the gap correspond to the case of the infinite negative scattering length. To study the BCS–BEC crossover, we have to solve the evolution equation for a wider range of the scattering lengths, including the positive values. The corresponding results for the evolution of the chemical potential as a function of the parameter $p_F a$ are shown in figure 2. While the vacuum scattering length is large and negative, the system is in the BCS-like phase with positive chemical potential, whereas if the scattering length is chosen to be large and positive reflecting the existence of a bound state near threshold the system ends up being the collection of weakly overlapping tightly bound pairs with negative chemical potential.

Now we turn to the results obtained with sharp cutoff (figure 3). One immediate observation is that the results become starting scale independent as long as $K > 5\,\mathrm{fm}^{-1}$ similar to the smooth regulator case. However, the phase transition occurs at lower values of the running scale $k \simeq 0.7\,\mathrm{fm}^{-1}$. At approximately $k \simeq 0.2\,\mathrm{fm}^{-1}$, the value of the gap becomes scale independent. It is worth mentioning that a sharp regulator can generate singularities in the evolution of the wavefunction renormalization parameters. However, unlike all the other wavefunction renormalization constants, the evolution of the boson wavefunction renormalization factor Z_ϕ can be obtained directly from the evolution of the effective potential (see equation (48)) for which a sharp regulator provides the smooth evolution without singularities. We expect some singularities to appear when calculating the fermion renormalization factors Z_ψ and Z_M. The gap evolutions obtained with the smooth and sharp regulators, being rather different at intermediate scales, approach each other with decreasing scale resulting in similar values for the physical gap. This is an encouraging result taking into account that, although the exact results must be independent of the choice of the regulator, in practice it is not guaranteed. The same conclusion also holds for other quantities. The couplings Z_ϕ and u_2 first grow with scale and then start decreasing eventually coming to zero. Chemical potential begins to decrease at the point of phase transition and becomes scale independent at $k \simeq 0.2\,\mathrm{fm}^{-1}$. However, in this case the numerical values of the chemical potentials obtained with different regulators differ by approximately 20% so that this quantity

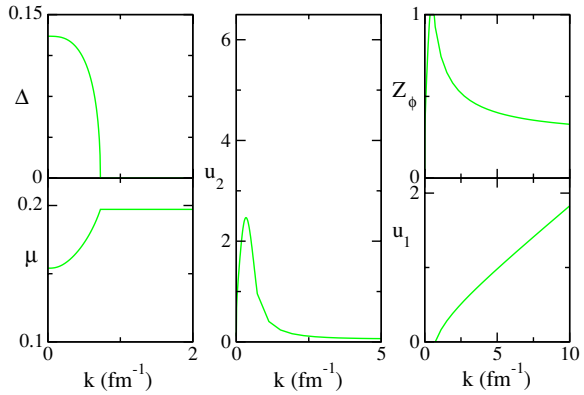

Figure 3. Evolution of the parameters when the sharp cutoff is used.

is more sensitive to the details of effective action and to the truncations made. We note that the sharp regulator can also describe the BCS–BEC crossover, although it gives somewhat smaller (negative) values of chemical potential at $p_F a > 1$. Applying our approach to neutron matter we find a gap comparable to ϵ_F, of the order of 30 MeV. There is a simple explanation for the smaller values (see [12, 13] or [14]). The argument can be given most succinctly for weak coupling, where the gap satisfies

$$\Delta = (8/\mathrm{e}^2)\epsilon_F \exp(-(\pi/2)\cot(\delta(p_F))). \tag{64}$$

For nucleon–nucleon scattering, $\cot\delta$ increases relatively quickly with momentum and the resulting reduction in the gap is substantial. We therefore expect that an extension of our approach to include the effective range should capture this physics. Indeed, if the 'in-medium' scattering length is identified with the Bethe–Goldstone G matrix calculated at zero energy but finite momenta [15] then we obtain the gap $\simeq 8$ MeV which is already compatible with the commonly accepted value. Of course, this is only a crude estimate and the proper calculations should be done using the momentum-dependent part of the four-fermion interaction.

In summary, the RG analysis for a many-fermion system with attraction leading to pairing effect has been carried out. It was found that, while at high scale the theory is in the symmetric phase, lowering the cutoff scale results in an appearance of a condensate and transition to a broken phase. Two different forms of the regulators were shown to lead to qualitatively similar results at the physical point. There are several points where our approach could be improved. We have already mentioned above the effective range effects. We should also include running of all the couplings and treat explicitly the particle–hole channels (RPA phonons) since these contain important physics. We would like to include the three-body force effects, which are required to satisfy the reparametrization invariance theorem [16] and possibly the long-range forces. As to further applications we plan to explore the superfluidity of the cold fermionic atoms in traps, temperature dependence of the BEC–BSC transitions and the formal relations the ERG with the other many-body approaches.

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